COMPUTATIONAL COMPLEXITY CLASSES AND ELEMENTARY RECURSIVE FUNCTIONS

LOUIS XIANG LI

This note is devoted to prove that the functions in the third level of Grzegorczyk hierarchy, indicated as $\mathcal{E}^2$, can be computed by a one-tape Turing Machine using linear space.

1. The third level of Grzegorczyk hierarchy $\mathcal{E}^2$

There is a intuitive understanding suggested by Grzegorczyk [1]. We first define projection function $I_i^n(x_1, x_2, ..., x_n)$ by $I_i^n(x_1, x_2, ..., x_n) = x_i$.

Now we define bounded recursion. A function $f \in F_{m+1}$ is said to be defined by **bounded recursion** via functions $g \in F_m, h \in F_{m+2}$, and $k \in F_{m+1}$ if, for any $x \in \mathbb{N}^m$ and $y \in \mathbb{N}$,

(1) f is defined by **primitive recursion** via g and h:
- $f(x, 0) = g(x)$,
- $f(x, y + 1) = h(x, y, f(x, y))$;

(2) f is bounded from above by k:
- $f(x, y) \leq k(x, y)$.

Now we define a sequence of computable function:

$$F_0(x, y) = y + 1$$

$$F_1(x, y) = x + y$$

$$F_2(x, y) = (x + 1) \cdot (y + 1)$$

$$\text{For } n \geq 2 : F_{n+1}(0, y) = F_n(x + 1, y + 1)$$

$$F_{n+1}(x + 1, y) = F_{n+1}(x, F_n(x, y))$$

We define Grzegorczyk classes $\mathcal{E}^n$ be the smallest class satisfies:

(1) Including $I_i^n(x_1, x_2, ..., x_n)$, $x + 1$, $F_n(x, y)$ as initial functions

(2) Closed under bounded recursion and substitution.

So we can conclude that the Grzegorczyk class $\mathcal{E}^2$ is the smallest class of functions contains $x+1, x\cdot y, I_i^n(x_1, x_2, ..., x_n)$ as initial functions and closed under bounded recursion and substitution.

2. Explaination of Minsky Machine

The Minsky machine is a multi-tape nonerasing Turing machine with a finite number of one-sided infinite to the right tapes whose terminal cells contain the symbol 1 and the other cells, 0. each tape is equipped by one reading head; at each step, the heads can, independently of each other, move by one cell to the left, to the right, or stay.

We say that a Minsky machine M which has at least n tapes computes an everywhere defined function $f(x_1, \ldots, x_n)$ if for any $x_1, \ldots, x_n$ the following conditions are fulfilled. If at the initial time the first n heads of the machine are at the cells numbered $x_1, \ldots, x_n$ (the terminal cells are numbered 0), while the other heads are at the terminal cells, then at the final time (when the machine M goes to the final state) the first head is at the cell numbered $f(x_1, \ldots, x_n)$.

The computation time (the number of steps) is denoted by $T_M(x_1, \ldots, x_n)$.

Let a machine M have s states $0, 1, \ldots, s-1$. We assume that 1 is the initial state, 0 is the final state. A programme of a k-tape Minsky machine M consists of commands of the form

$$e_1 \ldots e_k q \to d_1 \ldots d_k q',$$

where $e_1, \ldots, e_k \in \{0, 1\}$, $q, q' \in \{0, 1, \ldots, s-1\}$, $q \neq 0$, $d_1, \ldots, d_k \in \{-1, 0, 1\}$ and $d_i \neq -1$ if $e_i = 1$. What this command means is that if the machine M is in the state q at time t and the vector read by the heads is $(e_1, \ldots, e_k)$, then at time $t+1$ the machine M goes to the state q' and the head i, $1 \leq i \leq k$, moves by one cell to the left ($d_i = -1$), or to the right ($d_i = 1$), or stays put at the same cell before ($d_i = 0$).

By the configuration of a k-tape Minsky machine M at time t is meant the vector $(x_1, \ldots, x_k; q)$, where x_i is the number of the cell where the ith head is put, $1 \leq i \leq k$, and q is the state of the machine M at time t.

3. $\mathcal{E}^2$ = LINEAR SPACE

This result is one of the most important result in the presentation. To prove this we need to add some settings.

Theorem 3.1. *[3] The class $\mathcal{E}^2$ is the set of all functions computable by Minsky machines in a polynomial time.*

The proof of the theorem can be found in the cited paper.

Now we are going to prove our main theorem:

Theorem 3.2. $\mathcal{E}^2$ = *Linear space.*

To prove the theorem, we need to introduce some other settings:

We set

$$x \dot{-} y = \max\,(0, x - y)$$

And we define a function $[x/y]$ = the integer part of the number x/y if $y > 0$, and equal to zero if $y = 0$; Similarly, we define a function $[\log_2(x)]$ equal to the integer part of $\log_2(x)$ if $\log_2(x) > 0$, and 0 if $\log_2(x) = 0$.

We also set

$$\mathrm{rm}(x, y) = x \dot{-} y \cdot [x/y]$$

These functions can be easily shown in $\mathcal{E}^2$.

Definition 3.1. Let an everywhere defined function $f(x_1, ..., x_n)$ be computable by a Turing machine T. For fixed $x_1, ..., x_n$, the zone of computation of the value $f(x_1, ..., x_n)$ is the minimal connected part of the tape which contains the initial binary representation of the tuple $x_1, ..., x_n$ and all cells which the machine T visits during the computation of the value $f(x1, ..., xn)$ Let $S(l)$ be a function of a positive integer argument l with positive integer values. We say that a function $f(x_1, ..., x_n)$ is computable within the limits of the zone (or within the zone) $S(l)$ if for any tuple $x_1, ..., x_n$ with binary representation of length l the

zone of computing of the value $f(x_1...x_n)$ does not exceed $S(l)$, and we say a space is linear if $S(l) = al + b$, with $a, b \in \mathbb{N}$.

Now we will prove the main theorem:

Theorem 3.3. *[2]* $\mathcal{E}^2$ *= Linear Space*

Proof. To demonstrate that $\mathcal{E}^2$ is a subset of a Linear space, it's essential to show the conversion of a Minsky Machine to a one-tape Turing Machine. For a set of numbers $y_1, \ldots, y_k$ read by the Minsky machine's heads, we map them onto the Turing Machine's tape. The transformation of the set $y_1, \ldots, y_k$ to its immediate successor $y'_0, \ldots, y'_n$ by the Minsky Machine's transition function is straightforward.

Now, to prove $\mathcal{E}^2 \supset$ Linear space, we consider the conversion of a Turing machine, using linear space, into a Minsky Machine. Suppose at time t, the Turing machine's head is at cell s. Define L_s as the shortest word to the left of cell s containing all symbols 0 and 1 (if there are no such symbols, L_s is f by definition). Similarly, R_s is the word to the right of cell s, including it, containing all symbols 0 and 1. These words L_s and R_s are then converted into the corresponding numbers l_s and r_s in ternary notation.

If the Minsky machine's two tapes contain the numbers l_s and r_s, and the Turing machine moves to cell $s+1$ while writing symbol b in cell s, then the Minsky machine transforms l_s into $3l_s + b$ and r_s into $[s/3]$.

If the Turing machine's head moves left, the transformations are $l_{s-1} = [l_s/3]$ and $r_{s-1} = \mathrm{rm}(l_s, 3) + 3(r_s - \mathrm{rm}(r_s, 3) + b)$.

Initially, if the Turing machine's head is at cell s, next to the rightmost digit of x_n's binary representation, then $r_s = 0$. To compute l_s, convert the binary representations of $x_1, \ldots, x_n$ into ternary (replacing 0 with 1, and 1 with 2), and form the number with ternary representation $d(x_1)0d(x_2)0 \ldots 0d(x_n)$. This is easily achievable with a Minsky machine.

Finally, we consider the transformation of the Minsky machine's initial configuration into the pair of numbers coding the initial configuration of the Turing machine. The procedure involves computing binary digits of x_n and constructing the number whose ternary representation is $d(x_n)$, considering the contribution of each binary digit to $d(x_n)$. This is done

iteratively for each x_i, culminating in the representation of the initial state of the Turing machine.

In conclusion, the numbers like 2^i, 3^i used in this process are within a polynomial range of the variables $x_1, \ldots, x_n$.

□

References

[1] Andrzej Grzegorczyk. *Some classes of recursive functions*. eng. Warszawa: Instytut Matematyczny Polskiej Akademi Nauk, 1953. URL: http://eudml.org/doc/219317.

[2] S. S. Marchenkov. In: *Discrete Mathematics and Applications* 20.1 (2010), pp. 61–73. DOI: doi:10.1515/dma.2010.003. URL: https://doi.org/10.1515/dma.2010.003.

[3] S.S. Marchenkov. "The bases under superposition in some classes of recursive functions". In: *Mathematical Problems of Cybernetics* 3 (1991), pp. 115–139.